

MAKE YOUR BOY SURFACE

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ABSTRACT. This is an introductory article on the Boy surface. Boy found that $\mathbb{R}P^2$ can be immersed into \mathbb{R}^3 and published it 1901. (The image of) the immersion is called the Boy surface after Boy's discovery.

We have created a way to construct the Boy surface by using a pair of scissors, a piece of paper, and a strip of scotch tape. In this article we introduce the way.

1. INTRODUCTION

This is an introductory article on the Boy surface.

Boy discovered that $\mathbb{R}P^2$ can be immersed into \mathbb{R}^3 and published it 1901 in [1]. Boy is the name of a mathematician in Germany, May 4, 1879 – September 6, 1914. The immersion of $\mathbb{R}P^2$ into \mathbb{R}^3 which Boy found is called the Boy surface after his discovery. Note: The Boy surface is the name of (an image of) an immersion not that of a manifold.

See [3] for mathematical terms: $\mathbb{R}P^2$, immersion etc. P121 of [3] quotes [1].

[6] is the author's introductory book.

We have created a way to construct the Boy surface by using a pair of scissors, a piece of paper, and a strip of scotch tape. In this article we introduce the way.

In [2] Giller used the Boy surface. In [4, 5] the author cited [2] and used the Boy surface. It is his motivation to write this article.

We made a movie to explain our paper-craft. The website of the movie is connected with the author's website. You can find his website by typing in the author's name 'Eiji Ogasa' in the search engine. Click the indication under the title of this article in the part of the author's papers in his website. Then you see the website of the movie.

The author's website:

http://www.geocities.jp/n_dimension_n_dimension/list.html

The website of the movie:

http://www.geocities.jp/n_dimension_n_dimension/MakeyourBoysurface.html

Don't forget the three _ in each address.

2. PAPER-CRAFT

See Figure I, Figure II, and Figure III in the following two pages. Make three copies of Figure I, a copy of Figure II, and three copies of Figure III.

Note: Make the copies so that the length of the edge of each of the unit squares in Figure I is half of that in Figure II, III. If it might be difficult to take such a copy of Figure II (resp. III), then we recommend the following way: Take a copy of Figure I at first. After that, make Figure II, III on a paper by using a scale and a pencil.

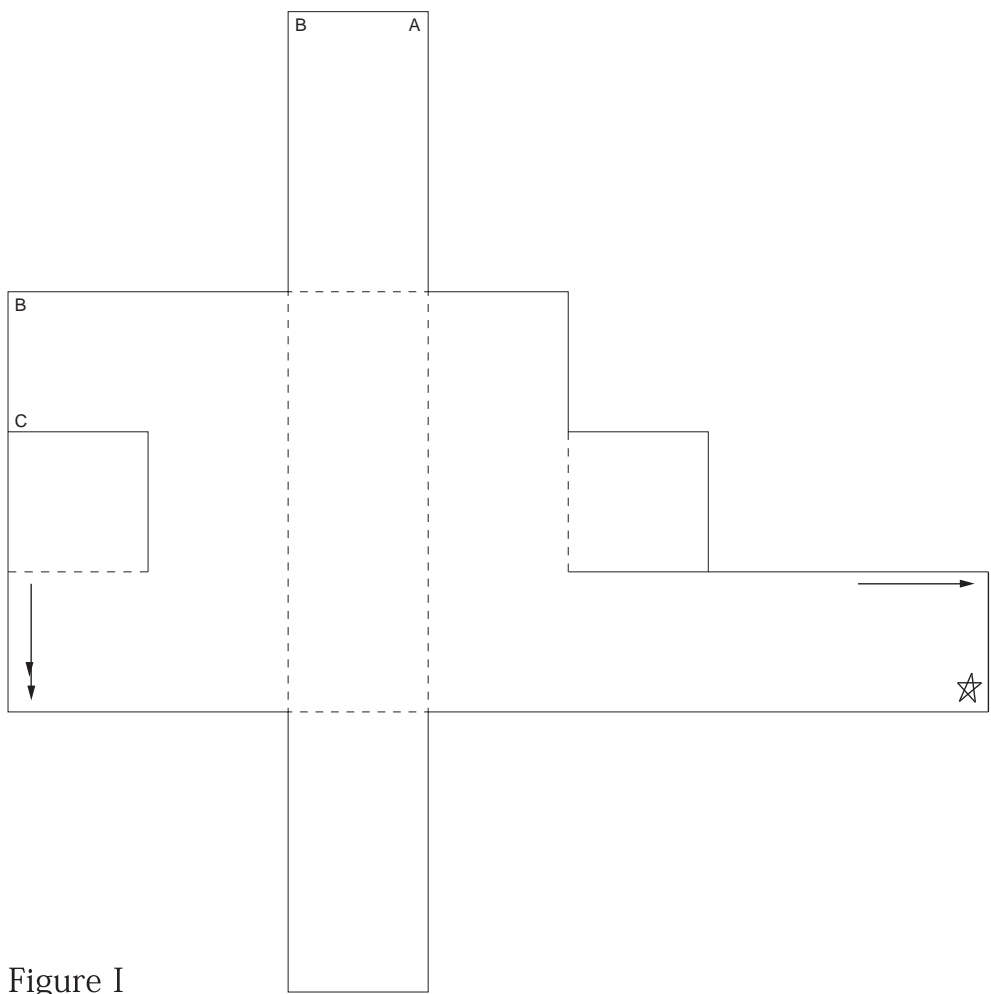


Figure I

Figure III

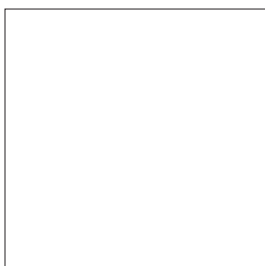
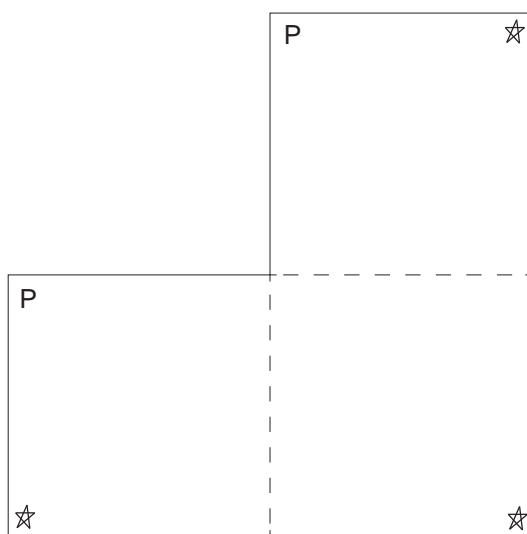


Figure II



Make Figure IV from the three copies of Figure III. What we obtain is called the piece IV. Note that we must cut the three copies of Figure III a few times. If necessarily, we divide one of the three copies into a few pieces once and after that, we make the piece IV from them by using a strip of scotch tape.

The piece IV is represented as follows if we take the xyz -axes . It is a union of
 $\{(x, y, z) | -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad z = 0\}$
and
 $\{(x, y, z) | -1 \leq y \leq 1, \quad -1 \leq z \leq 1, \quad x = 0\}$
and
 $\{(x, y, z) | -1 \leq z \leq 1, \quad -1 \leq x \leq 1, \quad y = 0\}.$

Take the following points in the piece IV as shown in Figure V.

$A(-1, 0, 0), \quad B(-1, 0, 1), \quad C(0, 0, 1)$
 $A'(0, -1, 0), \quad B'(1, -1, 0), \quad C'(1, 0, 0)$
 $A''(0, 0, -1), \quad B''(0, 1, -1), \quad C'''(0, 1, 0)$

We will use these points soon.

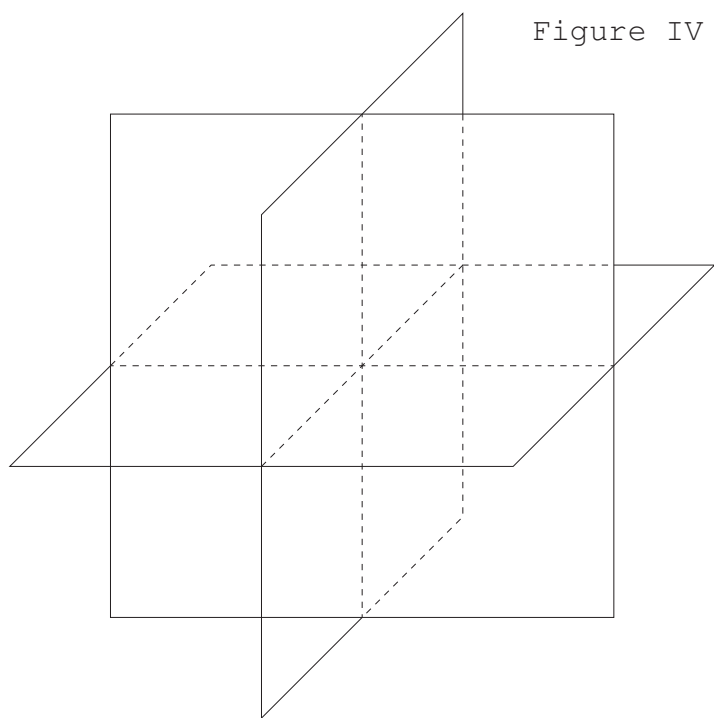


Figure IV

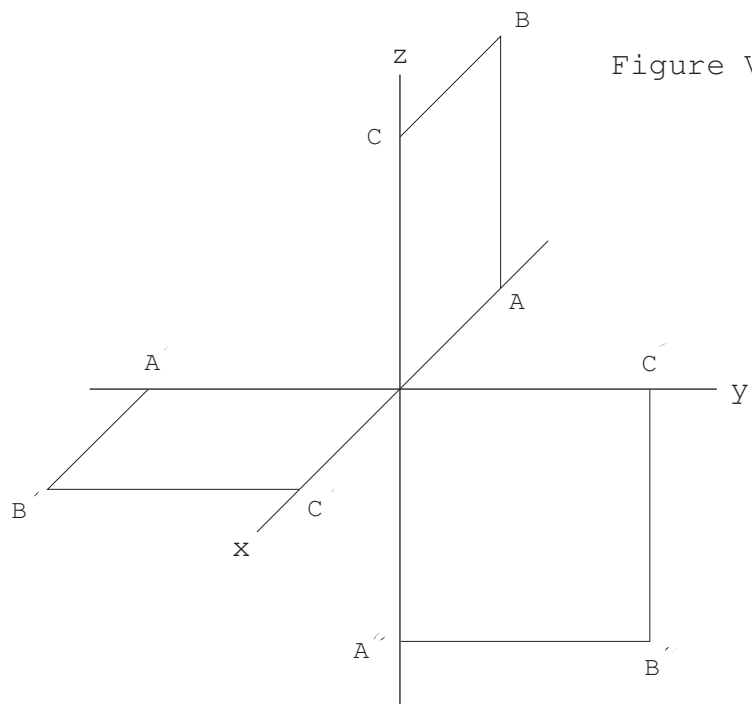
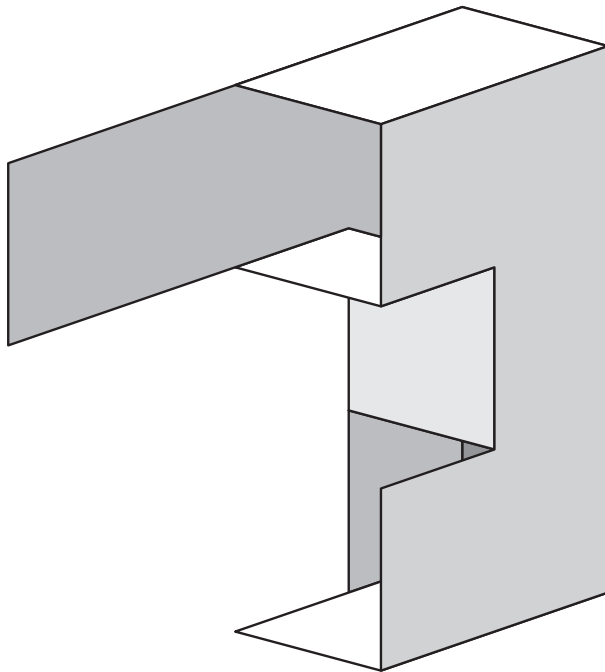


Figure V

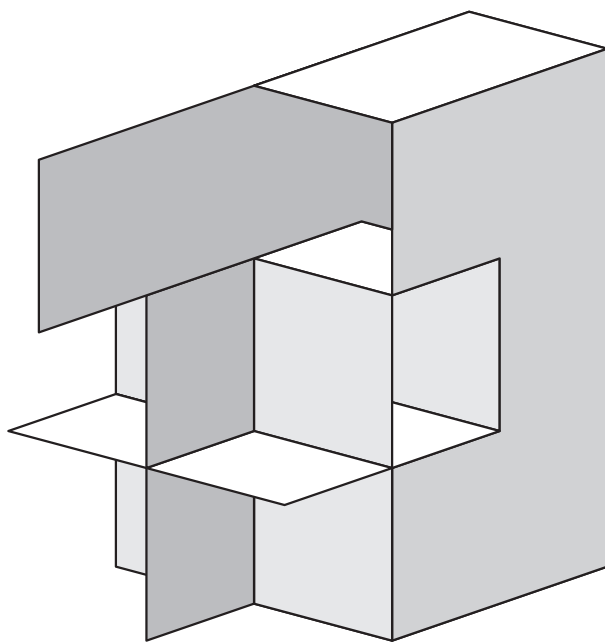
Cut each copy of Figure I along the solid lines. What we obtain after cutting is called the piece I.

Fold the piece I along the dotted line so that we see the dotted line inside, and make ‘the angle made by the paper at the dotted line’ 90° .

Use a strip of scotch tape and attach the edges which meet. Note that the two B meet. Then we obtain the following figure. It is called the piece I again.



We obtain three copies of the piece I.
 Call them the first piece I, the second piece I, the third piece I.
 The A, B, C is printed on each piece.
 Attach the first piece I to the piece IV with the following properties. A meets A . B
 meets B . C meets C .
 We obtain the following.

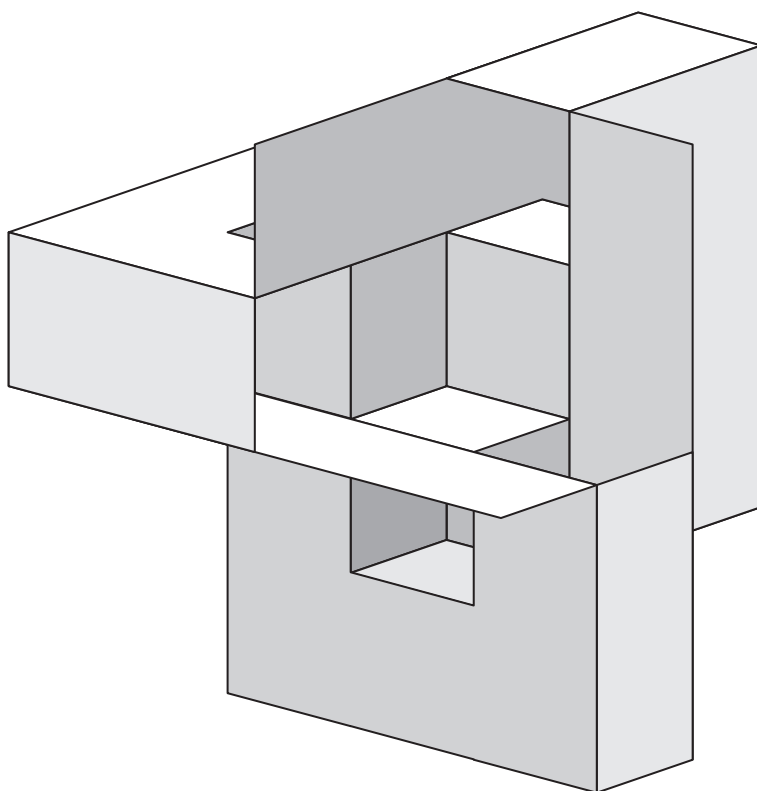


A, B, C in the second (resp. third) piece I are called A', B', C' (resp. A'', B'', C'').

Attach the second piece I to 'the piece IV with a copy of the piece I' with the following properties. A meets A' . B meets B' . C meets C' .

Attach the third piece I to 'the piece IV with two copies of the piece I' with the following properties. A'' meets A' . B'' meets B' . C'' meets C' .

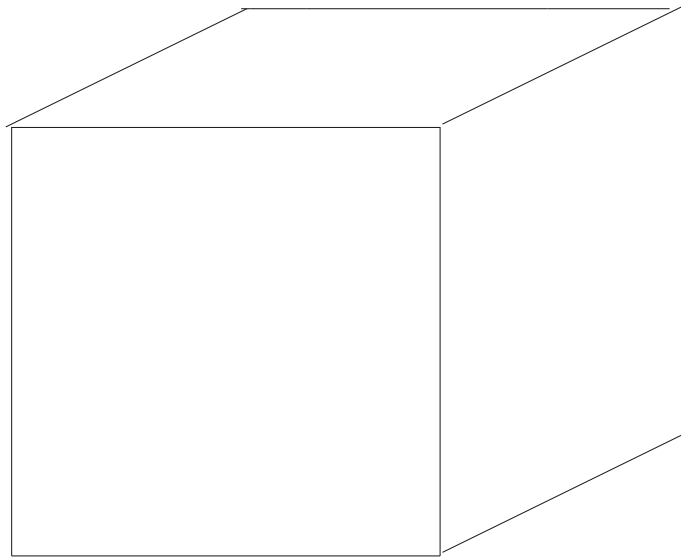
We obtain the following figure. It is called the piece VI. Note that the arrow in a copy of the piece I meets one of the 'double arrow' in another copy of the piece I.



Cut a copy of Figure II along the solid lines. What we obtain after cutting is called the piece II.

Fold the piece II along the dotted line so that we see the dotted line inside, and make ‘the angle made by the paper at the dotted line’ 90° .

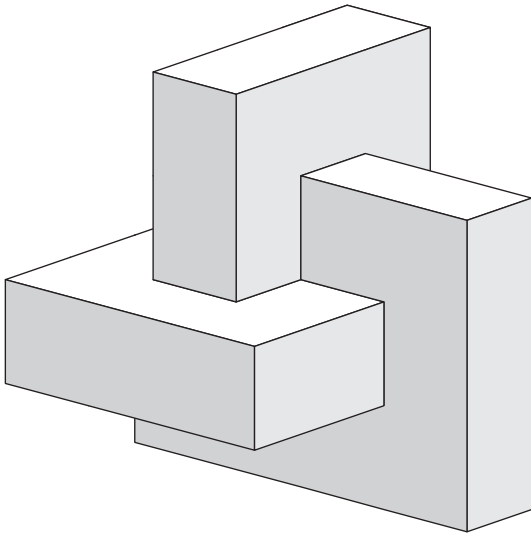
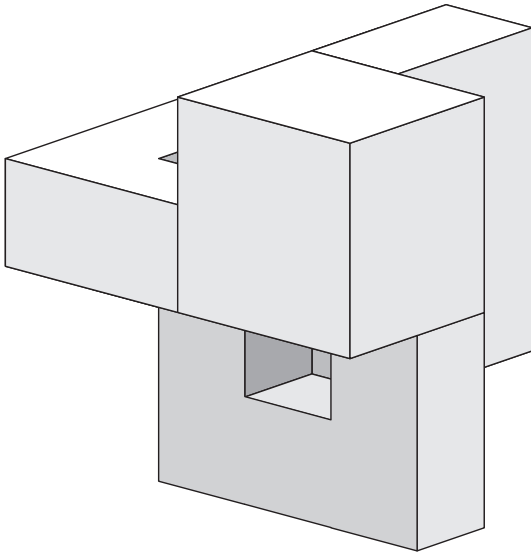
Use a strip of scotch tape and attach the edges which meet. Note that the two P meet. Then we obtain the following. It is called the piece II again.



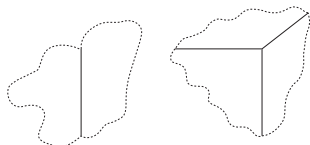
Piece II

Attach the piece II to the piece VI so that each star in the piece II meets each star in the piece VI. The result is the Boy surface.

We draw the two figures of the Boy surface, seeing from two different directions.

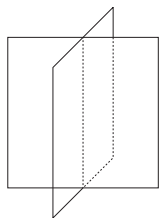


This Boy surface has the corner.

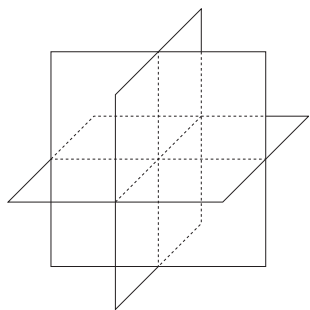


If you prefer the Boy surface without corner, imagine making the corner smooth. Or, make so.

The line which is the intersection of two sheets in the following figure is the set of double points. (The double point is the intersection of two sheets.)



The point $(0, 0, 0)$ in the piece IV is the triple point. (The triple point is the intersection of three sheets as shown below.) The Boy surface contains only one triple point.



It is known that we cannot immerse $\mathbb{R}P^2$ into \mathbb{R}^3 without a triple point.

3. PROVE

Prove that the paper-craft which we made is the immersion of $\mathbb{R}P^2$.

Sketchy proof: Calculate the homology groups or the betti number of the manifold whose immersion is the Boy surface. After that, use the Poincaré theorem on classifying surfaces.

Alternative sketchy proof: Remove the piece II and the piece IV from the Boy surface. Then the result is made from the three copies of the piece I. Remove the double point set by the following procedure.



Prove the result is (the Möbius band)–(three discs). Note that the boundary is a set of three circles. After that, use the Poincaré theorem on classifying surfaces.

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(Don't forget the three _ in this address. You can find this website by typing in the author's name 'Eiji Ogasa' in the search engine.)